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REPLY

Reply to comment on 'Integrable Kondo impurity in one-dimensional q-deformed t–J models'

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Abstract
This is a reply to the comment by P Schlottmann and A A Zvyagin.

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In their comment [1], Schlottmann and Zvyagin raise several issues regarding the nature of integrable impurities in one-dimensional quantum lattice models, and claim to expose false statements in our recent paper [2]. In order to address these issues in a pedagogical manner, we feel that it is appropriate to discuss these questions in terms of models based on the $gl(2|1)$ invariant solution of the Yang–Baxter equation. However, it is important from the outset to make it clear that the arguments we present below are general and apply to other classes of models.

The first point that we would like to make is that it is claimed in [1] that there are two approaches to the algebraic Bethe ansatz. However, it appears to us that approach (i) described in point (2) of [1] is the coordinate Bethe ansatz and we do not understand why this should be referred to as an algebraic Bethe ansatz. In the coordinate Bethe ansatz approach, one starts with a prescribed Hamiltonian and then solves the Schrödinger equation to obtain the two-particle scattering matrices and the particle–impurity scattering matrix. Together these scattering matrices form the monodromy matrix. It is impossible to infer the Hamiltonian from such a monodromy matrix. In this context we do not feel that point (1) of [1] answers our query about the existence of the impurity monodromy matrix in the algebraic approach (ii) of point (2) in [1]. Hereafter, we focus our attention on this case.

The solution of the Yang–Baxter equation

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v) \tag{1}$$

associated with the Lie superalgebra $gl(2|1)$ is an operator $R(u) \in \mathrm{End}\,(V \otimes V)$, where V is a three-dimensional $\mathbb{Z}_2$-graded space with one bosonic and two fermionic degrees of freedom. Explicitly, this operator takes the form

$$R(u) = u \cdot I \otimes I + P \tag{2}$$

where P is the $\mathbb{Z}_2$-graded permutation operator. For the purposes of constructing integrable one-dimensional quantum systems on a closed lattice, it is usual to introduce the Yang–Baxter algebra with elements $\hat{A}_{ij}(u), i, j = 1, 2, 3$. Putting these operators as the elements of a 3×3 matrix $A(u)$, the algebraic relations amongst these operators are determined by the requirement

$$R_{12}(u-v)A_1(u)A_2(v) = A_2(v)A_1(u)R_{12}(u-v). \tag{3}$$

It is apparent from equation (1) that the R-matrix (2) provides a representation of the Yang–Baxter algebra. The tensor product representation

$$T(u) = R_{0N}(u)\cdots R_{02}(u)R_{01}(u) \tag{4}$$

called the monodromy matrix, is also a representation of the Yang–Baxter algebra. The transfer matrix $t(u)$ is defined as the supertrace of the monodromy matrix, namely $t(u) = \mathrm{str}_0 T(u)$, and the Hamiltonian is the logarithmic derivative of the transfer matrix evaluated at $u = 0$, namely $H = t^{-1}(0)t'(0)$, where the prime indicates the derivative with respect to the spectral parameter u. For the present case this yields the supersymmetric t–J model as demonstrated in [3, 4].

In order to incorporate an integrable impurity into the model, one looks for a different representation of $A(u)$, say $L(u)$, and then constructs the impurity monodromy matrix

$$T(u) = R_{0N}(u)\cdots R_{02}(u)L_{01}(u-\theta) \tag{5}$$

with the transfer matrix and Hamiltonian defined as above. We agree with [1] that

- the position of the impurity Lax operator $L(u-\theta)$ in the chain is inconsequential (here we put it in the first site for convenience);
- the parameter θ, which plays the role of a coupling parameter, can be chosen arbitrarily;
- multiple impurity Lax operators can be placed in the monodromy matrix, each with an independent coupling θ_i, and in this case the positions of the impurity Lax operators have no effect on the Bethe ansatz equations.

We also agree with [1] that when one extends this approach to the case of an open chain, which is achieved by the additional requirement that there is a solution to the reflection equations, this situation persists. Any impurity Lax operator $L(u-\theta)$ for the closed chain is also an impurity Lax operator for the open chain, its position in the chain is immaterial regarding the Bethe ansatz solution, the coupling θ is arbitrary and the solution for the extension to multiple impurities with independent couplings is independent of the position of the impurities in the chain. The essential feature, which seems to have been missed in [1], is that the converse is not necessarily true. There are classes of boundary impurities, which are obtained via solutions of the reflection equations, which have no analogue in the corresponding closed chain case. This is one of the primary results that we have tried to convey in [2] for the q-deformed t–J model. The case of the supersymmetric t–J model (with $q = 1$) has been studied in [5], which is relevant to our discussions here. Our results show the existence of boundary Kondo impurities for the supersymmetric t–J model which have no analogue in the closed chain case. We return to this point later.

Next we look, in closer detail, at the types of impurity Lax operators $L(u)$ which exist for the present case of the supersymmetric t–J model. We do not claim that the present classification is complete. One class takes the generic form

$$L(u) = u \cdot I \otimes I + \sum_{i,j}^{3} (-1)^{[i]} e_i^j \otimes \pi\left(E_j^i\right) \tag{6}$$

where e_i^j is the matrix acting on V with 1 in the (i, j) position and zeroes elsewhere. π is an arbitrary representation of the $gl(2|1)$ generators E_j^i satisfying the commutation relations

$$\left[E_j^i, E_l^k\right] = \delta_j^k E_l^i - (-1)^{([i]+[j])([k]+[l])}\delta_l^i E_j^k.$$

Above, $[i]$ is the grading index so that $[i] = 0$ if i is a bosonic label and $[i] = 1$ if i is a fermionic label. Below we adopt the convention $[1] = [2] = 1, [3] = 0$. Choosing the fundamental representation in equation (6) for the $gl(2|1)$ generators with $\theta = 0$ yields the R-matrix (2). The case of non-zero θ gives rise to an impurity Lax operator which has been studied in [6]. Other impurity Lax operators can be obtained by choosing different representations for $gl(2|1)$. For the case of the dual to the fundamental representation, this yields the impurity model studied in [7, 8]. Choosing the four-dimensional representations yields the impurity model of [9, 10]. According to [1], these models belong to class (ii) described in point (2), in which there are two levels of Bethe ansatz equations corresponding to charge and spin degrees of freedom. However, it is claimed in point (4) of [1] that 'the effect of the impurity matrix in the charge sector (first level Bethe ansatz) changes the commutation relations in the spin sector (second level)'. This statement conflicts with the explicit studies of the Bethe ansatz solutions conducted in [7–11] and is also not supported by the findings of [12] where the algebraic Bethe ansatz was performed for the $gl(2|1)$ invariant R-matrix in the most general context, with the only assumption made being the existence of a reference state on which the action of the monodromy matrix takes an upper triangular form. This raises serious doubt over the validity of the Bethe ansatz solution presented in appendix A of [13], which we have already discussed in the introduction of [2]. Some concerns about the Bethe ansatz treatments of one of the authors of [1] have been raised by other researchers [14–19] and we believe that a similar lack of appreciation of the subtleties involved has caused the current misunderstanding regarding our work. Furthermore, in point (5) of [1] the authors attempt to counter our criticisms by claiming that the operator $\hat{A}_{21}(u)$ is a 'raising operator'. Appendix A of [13] claims that the operators $\hat{A}_{12}(u)$, $\hat{A}_{13}(u)$, $\hat{A}_{21}(u)$ and $\hat{A}_{23}(u)$ all vanish on the reference state. If $\hat{A}_{21}(u)$ is a raising operator then $\hat{A}_{12}(u)$, $\hat{A}_{13}(u)$ and $\hat{A}_{23}(u)$ are lowering operators. The fact that they each vanish on the reference state indicates that the reference state is a 'lowest weight state', rather than a 'highest weight state'. Clearly, a proper treatment does not assume that the raising operator $\hat{A}_{21}(u)$ vanishes on a lowest weight state. These assumptions were not made in the Bethe ansatz treatments of [7, 8, 10, 11] ([9] is an exception where such an assumption is valid) and indeed [12] makes a detailed discussion about this very issue in the general context.

We also believe that this assumption of [13] is inconsistent with the transfer matrix eigenvalue (A1) presented therein. Here we give our argument in detail. Let $|\Psi\rangle$ denote the reference state for the algebraic Bethe ansatz procedure. It is assumed in [13] that the reference state is an eigenstate of the diagonal operators $\hat{A}_{ii}(u)$, namely

$$\hat{A}_{ii}(u)|\Psi\rangle = \lambda_i(u)|\Psi\rangle \tag{7}$$

and

$$\hat{A}_{21}(u)|\Psi\rangle = \hat{A}_{12}(u)|\Psi\rangle = 0. \tag{8}$$

The explicit commutation relations between the operators $\hat{A}_{ij}(u), i, j = 1, 2$ can be conveniently expressed as [21]

$$(u - v)[\hat{A}_{ij}(u), \hat{A}_{kl}(v)] = \hat{A}_{il}(u)\hat{A}_{kj}(v) - \hat{A}_{il}(v)\hat{A}_{kj}(u)$$

and in particular we have

$$(u - v)[\hat{A}_{12}(u), \hat{A}_{21}(v)] = \hat{A}_{11}(u)\hat{A}_{22}(v) - \hat{A}_{11}(v)\hat{A}_{22}(u). \tag{9}$$

With equation (9) acting on $|\Psi\rangle$, this immediately shows, through the use of equations (7) and (8), that

$$\lambda_1(u)\lambda_2(v) = \lambda_1(v)\lambda_2(u)$$

for all values of the parameters u and v, which immediately implies that

$$\lambda_2(u) = c\lambda_1(u)$$

for some constant c independent of u. The constant will be unity if there is spin reflection symmetry. The action of the transfer matrix $t(u)$ on $|\Psi\rangle$ thus reads

$$t(u)|\Psi\rangle = (\lambda_3(u) - (1+c)\lambda_1(u))|\Psi\rangle$$

which is inconsistent with the result obtained by setting $M = N = 0$ in (A1) of [13].

To further confuse the issue, point (5) of [1] mistakenly identifies the parameter l introduced in the introduction of [2] as characterizing the symmetry of the impurity Lax operator. It is in fact a label for a class of atypical representations of the $gl(2|1)$ superalgebra. Applying such a representation to equation (6) produces an impurity Lax operator which bears a resemblance to that alluded to in appendix B of [13]. It should be clear in this instance why the operator $\hat{A}_{21}(u)$ cannot vanish on the (lowest weight) reference state, since $\hat{A}_{21}(u)$ is a spin raising operator, and the 'higher spin' atypical representations of $gl(2|1)$, which are used for the local impurity Hilbert space, do not admit a spin singlet. A description and a discussion of the applicability of these representations are given in [20].

Point (6) of [1] provides a misrepresentation of our claims. We agree that it is 'incorrect that impurities can only modify the first level Bethe ansatz equations'. The point that we make in the introduction of [2] is that if $\hat{A}_{12}(u)$ and $\hat{A}_{21}(u)$ both vanish on the reference state (as is assumed in [13]), then the second level Bethe ansatz equations are unchanged (contradicting the final results presented in [13]).

We now return to the issue of impurity Lax operators. Another class can be obtained by taking the Lax operator associated with a subalgebra of the symmetry of the R-matrix. For example, in the present $gl(2|1)$ invariant case there is a natural $gl(2)$ subalgebra which has the Lax operator

$$L(u) = u \cdot \left(e_1^1 + e_2^2\right) \otimes I - \sum_{i,j}^{2} e_i^j \otimes \pi\left(F_j^i\right) \tag{10}$$

where the operators F_j^i satisfy the $gl(2)$ commutation relations. This impurity operator can be used in equation (5) to construct an impurity model which would appear to describe a Kondo-type impurity, except for one important facet. The operator (10) is singular, which in turn makes the transfer matrix singular, and does not yield a well-defined Hamiltonian in terms of the logarithmic derivative of the transfer matrix. It is for this reason that we have looked to the reflection equations for the study of Kondo impurities in the t–J model in [5].

Finally, we turn to the pertinent results of [2, 5]. In these papers, we have solved the reflection equations for the purpose of introducing Kondo impurities in the supersymmetric t–J model and the q-deformed version, in a fashion which permits a well-defined Hamiltonian. Using this approach, the impurity interaction is purely of the Kondo-type in the sense that there are no charge degrees of freedom, in spite of the protestations of [13] that this situation is not possible. The argument of [13] that charge degrees of freedom should be present for the impurity site, on the basis of symmetry, is wrong because it is possible to break symmetry at the boundary and maintain integrability, just as in the case of boundary scalar fields (see [22] for a discussion of this point in relation to integrable models with broken $U_q(gl(n))$ symmetry due to boundary interactions). In [5], the symmetry of the Hamiltonian is broken from $gl(2|1)$ to $gl(2)$ due to the inclusion of the boundary interactions.

Solutions of the reflection equation are specific to the construction of open chain models and there is no justification to expect that such integrable boundary impurities have an analogue in the bulk case. An attempt to do so for the model of [5] yields a singular open chain transfer matrix analogous to that discussed above for the periodic case, for which the Hamiltonian cannot be defined. We have argued vehemently in the conclusion of [5] that the boundary impurity K-matrices are operator valued and do not arise as the 'dressing' of a scalar K-matrix.

This indicates that these impurities are integrable only when situated on the boundary and do not have a bulk counterpart. (The situation really is no different to the case of integrable boundary scalar fields which also cannot be moved into the bulk. This is also true of model I in [23] studying a spin ladder model with a defect in the rung coupling, where integrability holds only when the defect is on the boundary.) It is important to emphasize that our construction is entirely different from that discussed in [24] where the K-matrix is 'the ordinary c-number reflection matrix of a free boundary sandwiched between two forward scattering impurity matrices'. This conclusion has been confirmed by [25] who developed the 'projection method' for the construction of such impurities in a general context. Application to the specific case of the t–J model has been studied in [26]. Furthermore, the case of boundary impurities in the q-deformed t–J model has been independently studied in [27, 28] and also confirms the validity of our approach.

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